A_p-A_∞ ESTIMATES FOR GENERAL MULTILINEAR SPARSE OPERATORS

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Abstract. In this paper, we study the A_p-A_∞ estimates for a class of multilinear dyadic positive operators. As applications, the A_p-A_∞ estimates for different operators e.g. multilinear square functions and multilinear Fourier multipliers can be deduced very easily.

1. Introduction

The weighted norm inequality is a hot topic in harmonic analysis. In 1980s, Buckley [1] studied the quantitative relation between the weighted bound of Hardy-Littlewood maximal function and the A_p constant. Specifically, he showed that

$$\|M\|_{L^p(w)} \le c[w]_{A_p}^{\frac{1}{p-1}},$$

where recall that

$$[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle w^{1-p'}\rangle_Q^{p-1}.$$

Here and through out, $\langle\cdot\rangle_Q$ denotes the average over Q.

Since then, the sharp weighted estimates for Calderón-Zygmund operators has attracted many authors' interest, which was referred to as the famous A_2 conjecture. The A_2 conjecture (now theorem) asserts that

$$\|T\|_{L^p(w)} \le c[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}.$$

It was finally proved by Hytönen [11]. The interested readers can consult [12] for a survey on the history of the different proofs given for A_2 theorem. Moreover, Hytönen and Lacey [13] extends the A_2 theorem to the so-called A_p-A_∞ type estimates, i.e.,

$$\|T(\cdot\sigma)\|_{L^p(\sigma)\to L^p(w)} \le c[w,\sigma]_{A_p}^{\frac{1}{p}}([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}),$$

where

$$[w,\sigma]_{A_p} := \sup_Q \langle w\rangle_Q \langle \sigma\rangle_Q^{p-1},\ [w]_{A_\infty} := \sup_Q \frac{1}{w(Q)}\int_Q M(\mathbf{1}_Q w)dx$$

and σ needn't to be the dual weight of w, i.e., we don't require that $\sigma = w^{1-p'}$.

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Now the story goes to the multilinear case. First we need to extend the A_p weights to the multilinear case. Let $1 < p_1, \dots, p_m < \infty$ and p be numbers such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and denote $\vec{P} = (p_1, \dots, p_m)$. Now we define $[w, \vec{\sigma}]_{A_{\vec{P}}}$ constant:

$$[w, \vec{\sigma}]_{A_{\vec{P}}} = \sup_Q \langle w \rangle_Q \prod_{i=1}^m \langle \sigma_i \rangle_Q^{\frac{p}{p_i'}}.$$

In the one weight case, i.e., $\sigma_i = w_i^{1-p_i'}$ and $w = \prod_{i=1}^m w_i^{p/p_i}$, we say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if $[w, \vec{\sigma}]_{A_{\vec{P}}} < \infty$, see [22]. For the Buckley type estimate, the second author, Moen and Sun [23] studied the sharp weighted estimates for multilinear maximal operators for all indices and multilinear Calderón-Zygmund operators when $p > 1$. The corresponding A_p-A_∞ estimate was obtained in [7] and [24], respectively. Specifically, the result for multilinear maximal operators reads as

$$\|\mathcal{M}(\cdot \vec{\sigma})\|_{L^{p_1}(\sigma_1) \times \cdots \times L^{p_m}(\sigma_m) \to L^p(w)} \le [w, \vec{\sigma}]_{A_{\vec{P}}}^{\frac{1}{p}} \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\frac{1}{p_i}}.$$

As to the multilinear Calderón-Zygmund operators, if $p > 1$, then

$$\|T(\cdot \vec{\sigma})\|_{L^{p_1}(\sigma_1) \times \cdots \times L^{p_m}(\sigma_m) \to L^p(w)} \le [w, \vec{\sigma}]_{A_{\vec{P}}}^{\frac{1}{p}} \Big(\prod_{i=1}^m [\sigma_i]_{A_\infty}^{\frac{1}{p_i}} + [w]_{A_\infty}^{\frac{1}{p'}} (\sum_{j=1}^m \prod_{i \ne j} [\sigma_i]_{A_\infty}^{\frac{1}{p_i}}) \Big).$$

The spirit of the above results is reducing the problem to consider the so-called sparse operators. Recall that given a dyadic grid $\mathcal{D}$, we say a collection $\mathcal{S} \subset \mathcal{D}$ is sparse if

$$\Big| \bigcup_{\substack{Q' \subsetneq Q \\ Q', Q \in \mathcal{S}}} Q' \Big| \le \frac{1}{2} |Q|,$$

and we denote $E_Q := Q \setminus \cup_{Q' \in \mathcal{S}, Q' \subsetneq Q} Q'$. Now given a sparse family $\mathcal{S}$ over a dyadic grid D and $\gamma \ge 1$, a *general multilinear sparse operator* is an averaging operator over $\mathcal{S}$ of the form

$$T_{p_0, \gamma, \mathcal{S}}(\vec{f})(x) = \left(\sum_{Q \in S} \left[\prod_{i=1}^m \langle f_i \rangle_{Q, p_0} \right]^\gamma \chi_Q(x) \right)^{1/\gamma}$$

where $p_0 \in [1, \infty)$ and for any cube Q,

$$\langle f \rangle_{Q, p_0} := \left(\frac{1}{|Q|} \int_Q |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}}.$$

It was proved in [5] that the multilinear Calderón-Zygmund operators are dominated pointwisely by $T_{1,1,\mathcal{S}}$. In [4], Bui and the first author also showed that the multilinear square functions are dominated pointwisely by $T_{1,2,\mathcal{S}}$, and therefore, they obtained the Buckley type estimate for multilinear square functions. For $\gamma = 1$ and general p_0, it was shown in [2] that $T_{p_0,1,\mathcal{S}}$ can dominates a large class of operators with rough kernels (which include multilinear Fourier multipliers) as well. Therefore, everything are reduced to study $T_{p_0,\gamma,\mathcal{S}}$. Our main result states as follows.

Theorem 1.1. *Let $\gamma > 0$. Suppose that $p_0 < p_1, \dots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let w and $\vec{\sigma}$ be weights satisfying that $[w, \vec{\sigma}]_{A_{\vec{P}/p_0}} < \infty$ and $w, \sigma_i \in A_\infty$ for $i = 1, \dots, m$. If $\gamma \geq p_0$, then*

$$\left\| T_{p_0,\gamma,\mathcal{S}}(\vec{f}) \right\|_{L^p(w)} \lesssim [w, \vec{\sigma}]_{A_{\vec{P}/p_0}}^{\frac{1}{p}} \Big(\prod_{i=1}^{m} [\sigma_i]_{A_\infty}^{\frac{1}{p_i}} + [w]_{A_\infty}^{(\frac{1}{\gamma} - \frac{1}{p})_+} \sum_{j=1}^{m} \prod_{i \neq j} [\sigma_i]_{A_\infty}^{\frac{1}{p_i}} \Big) \times \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)},$$

where $w_i = \sigma_i^{1 - \frac{p_i}{p_0}}$, $i = 1, \cdots, m$ and

$$\left(\frac{1}{\gamma} - \frac{1}{p} \right)_+ := \max \left\{ \frac{1}{\gamma} - \frac{1}{p}, 0 \right\}.$$

If $\gamma < p_0$, then the above result still holds for all $p > \gamma$.

The proof of Theorem 1.1 is quite technical. In the literature, the A_p-A_∞ estimates usually follows from testing condition. Our technique provide a way to obtain A_p-A_∞ estimates without testing conditions. The idea follows from a recent paper by Lacey and the second author [15], where they studied the A_p-A_∞ estimates for square functions in the linear case. We generalize their method to suit for the multilinear case with general parameters γ and p_0.

2. Proof of Theorem 1.1

Let us first observe that it suffices to prove Theorem 1.1 for $p_0 = 1$. Indeed, suppose Theorem 1.1 holds for $p_0 = 1$. Consider the two weight norm inequality

$$\|T_{p_0,\gamma,\mathcal{S}}(f,g)\|_{L^p(w)} \leq \mathcal{N} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)}, \tag{2.1}$$

where we use $\mathcal{N}$ to denote the best constant such that (2.1) holds. Rewrite (2.1) as

$$\|T_{p_0,\gamma,\mathcal{S}}(f^{1/p_0}, g^{1/p_0})\|_{L^p(w)}^{p_0} \leq \mathcal{N}^{p_0} \|f^{1/p_0}\|_{L^{p_1}(w_1)}^{p_0} \|g^{1/p_0}\|_{L^{p_2}(w_2)}^{p_0},$$

which is equivalent to the following

$$\|T_{1,\frac{\gamma}{p_0},\mathcal{S}}(f,g)\|_{L^{p/p_0}(w)} \leq \mathcal{N}^{p_0} \|f\|_{L^{p_1/p_0}(w_1)} \|g\|_{L^{p_2/p_0}(w_2)}.$$

Then by our assumption, we have

$$\mathcal{N} \lesssim [w, \vec{\sigma}]_{A_{\vec{P}/p_0}}^{\frac{1}{p}} \Big([\sigma_1]_{A_\infty}^{\frac{1}{p_1}} [\sigma_2]_{A_\infty}^{\frac{1}{p_2}} + [w]_{A_\infty}^{(\frac{1}{\gamma} - \frac{1}{p})_+} ([\sigma_1]_{A_\infty}^{\frac{1}{p_1}} + [\sigma_2]_{A_\infty}^{\frac{1}{p_2}}) \Big).$$

So we concentrate on the case $p_0 = 1$. As in [24], we begin with $m = 2$, that is we deal with the dyadic bilinear operators:

$$T(f,g) := \Big(\sum_{Q \in \mathcal{S}} \langle f \rangle_Q^\gamma \langle g \rangle_Q^\gamma \mathbf{1}_Q \Big)^{\frac{1}{\gamma}}$$

and we shall give the corresponding A_p-A_∞ estimate.

Without loss of generality, we can assume that all cubes in $\mathcal{S}$ are contained in some root cube. As usual we only work on a subfamily $\mathcal{S}_a$, which is defined by the following

$$\mathcal{S}_a := \{Q \in \mathcal{S} : 2^a < \langle w \rangle_Q \langle \sigma_1 \rangle_Q^{\frac{p}{p_1'}} \langle \sigma_2 \rangle_Q^{\frac{p}{p_2'}} \leq 2^{a+1}\}.$$

Now we can define the principal cubes $\mathcal{F}$ for (f,σ_1) and $\mathcal{G}$ for (g,σ_2). Namely,

$$\begin{aligned}\mathcal{F} &:= \bigcup_{k=0}^{\infty}\mathcal{F}_k, \quad \mathcal{F}_0 := \{\text{maximal cubes in } \mathcal{S}_a\}\\ \mathcal{F}_{k+1} &:= \bigcup_{F\in\mathcal{F}_k}\mathrm{ch}_{\mathcal{F}}(\mathrm{F}), \quad \mathrm{ch}_{\mathcal{F}}(\mathrm{F}) := \{\mathrm{Q}\subsetneq \mathrm{F}\,\text{maximal s.t.}\langle \mathrm{f}\rangle_{\mathrm{Q}}^{\sigma_1} > 2\langle \mathrm{f}\rangle_{\mathrm{F}}^{\sigma_1}\},\end{aligned}$$

and analogously for $\mathcal{G}$. We use $\pi_{\mathcal{F}}(Q)$ to denote the minimal cube in $\mathcal{F}$ which contains Q and $\pi(Q)=(F,G)$ means that $\pi_{\mathcal{F}}(Q)=F$ and $\pi_{\mathcal{G}}(Q)=G$. By our construction, it is easy to see that

$$\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^{p_1}\sigma_1(F)\lesssim \|f\|_{L^{p_1}(\sigma_1)}^{p_1}. \tag{2.2}$$

We are going to prove that if w,σ_1,σ_2 be weights satisfying that $[w,\vec{\sigma}]_{A_{\vec{P}}}<\infty$ and $w,\sigma_1,\sigma_2\in A_\infty$. Then

$$\begin{aligned}\|T(f\sigma_1,g\sigma_2)\|_{L^p(w)} \lesssim [w,\vec{\sigma}]_{A_{\vec{P}}}^{\frac{1}{p}}\Big([\sigma_1]_{A_\infty}^{\frac{1}{p_1}}[\sigma_2]_{A_\infty}^{\frac{1}{p_2}}\\ + [w]_{A_\infty}^{(\frac{1}{\gamma}-\frac{1}{p})_+}([\sigma_1]_{A_\infty}^{\frac{1}{p_1}}+[\sigma_2]_{A_\infty}^{\frac{1}{p_2}})\Big)\|f\|_{L^{p_1}(\sigma_1)}\|g\|_{L^{p_2}(\sigma_2)}.\end{aligned}$$

First, we consider the case $p\le\gamma$ with $\gamma\ge 1$.

2.1. **The case $p\le\gamma$ with $\gamma\ge 1$.** In this case, we have

$$\begin{aligned}&\Big\|\Big(\sum_{Q\in\mathcal{S}_a}\langle f\sigma_1\rangle_Q^\gamma\langle g\sigma_2\rangle_Q^\gamma\mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}\\ &=\Big\|\Big(\sum_{Q\in\mathcal{S}_a}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma\mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}\\ &\lesssim\Big\|\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^\gamma\sum_{G\in\mathcal{G}}(\langle g\rangle_G^{\sigma_2})^\gamma\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma\mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}\\ &\le\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^p\sum_{G\in\mathcal{G}}(\langle g\rangle_G^{\sigma_2})^p\Big\|\Big(\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma\mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}^p\Big)^{\frac{1}{p}}\\ &\lesssim\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^p\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}(\langle g\rangle_G^{\sigma_2})^p\Big\|\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q\mathbf{1}_Q\Big\|_{L^p(w)}^p\Big)^{\frac{1}{p}}\\ &+\Big(\sum_{G\in\mathcal{G}}(\langle g\rangle_G^{\sigma_2})^p\sum_{\substack{F\in\mathcal{F}\\ F\subset G}}(\langle f\rangle_F^{\sigma_1})^p\Big\|\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q\mathbf{1}_Q\Big\|_{L^p(w)}^p\Big)^{\frac{1}{p}}\\ &:= I+II.\end{aligned}$$

By symmetry we only focus on estimating I. From [24], we already know that

$$\begin{aligned}(2.3)\qquad &\Big\| \sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}} \langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q\mathbf{1}_Q\Big\|_{L^p(w)}\\ &\lesssim 2^{\frac{a}{p}}\Big(\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\sigma_1(Q)\Big)^{\frac{1}{p_1}}\Big(\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\sigma_2(Q)\Big)^{\frac{1}{p_2}}.\end{aligned}$$

We also recall a fact that, for $\sigma\in A_\infty$ and $\mathcal{S}$ a sparse family, we have

$$\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}}\sigma(Q)\le 2\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}}\langle\sigma\rangle_Q|E_Q|\le 2\int_R M(\mathbf{1}_R\sigma)dx\le 2[\sigma]_{A_\infty}\sigma(R).$$

Therefore,

$$\begin{aligned}I&\lesssim 2^{\frac{a}{p}}\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^p\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}(\langle g\rangle_G^{\sigma_2})^p\Big(\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\sigma_1(Q)\Big)^{\frac{p}{p_1}}\Big(\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\sigma_2(Q)\Big)^{\frac{p}{p_2}}\Big)^{\frac{1}{p}}\\
&\lesssim 2^{\frac{a}{p}}[\sigma_2]_{A_\infty}^{\frac{1}{p_2}}\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^p\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}(\langle g\rangle_G^{\sigma_2})^p\Big(\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\sigma_1(Q)\Big)^{\frac{p}{p_1}}\sigma_2(G)^{\frac{p}{p_2}}\Big)^{\frac{1}{p}}\\
&\le 2^{\frac{a}{p}}[\sigma_2]_{A_\infty}^{\frac{1}{p_2}}\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^p\Big(\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}(\langle g\rangle_G^{\sigma_2})^{p_2}\sigma_2(G)\Big)^{\frac{p}{p_2}}\Big(\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\sigma_1(Q)\Big)^{\frac{p}{p_1}}\Big)^{\frac{1}{p}}\\
&\le 2^{\frac{a}{p}}[\sigma_2]_{A_\infty}^{\frac{1}{p_2}}\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^{p_1}\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G)}}\sigma_1(Q)\Big)^{\frac{1}{p_1}}\\
&\times\Big(\sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}(\langle g\rangle_G^{\sigma_2})^{p_2}\sigma_2(G)\Big)^{\frac{1}{p_2}}\\
&\lesssim 2^{\frac{a}{p}}[\sigma_1]_{A_\infty}^{\frac{1}{p_1}}[\sigma_2]_{A_\infty}^{\frac{1}{p_2}}\|f\|_{L^{p_1}(\sigma_1)}\|g\|_{L^{p_2}(\sigma_2)},\end{aligned}$$

where (2.2) is used in the last step.

2.2. **The case $p>\gamma$ with $p_1=\max\{p_1,p_2,q'\}$.** Here $q=p/\gamma$. By duality, we have

$$\begin{aligned}&\Big\|\Big(\sum_{Q\in\mathcal{S}_a}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma\mathbf{1}_Q\Big)^{\frac{1}{\gamma}}\Big\|_{L^p(w)}^\gamma\\ &=\sup_{\|h\|_{L^{q'}(w)}=1}\sum_{Q\in\mathcal{S}_a}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q).\end{aligned}$$

Now we suppress the supremum, and denote by $\mathcal{H}$ the principal cubes associated to (h,w). Similarly, $\pi(Q)=(F,G,H)$ means that $\pi_{\mathcal{F}}(Q)=F$, $\pi_{\mathcal{G}}(Q)=G$ and $\pi_{\mathcal{H}}(Q)=H$. We

have

$$
\begin{aligned}
&\sum_{Q\in\mathcal{S}_a}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)\\
&=\sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}\sum_{\substack{H\in\mathcal{H}\\ H\subset G}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)\\
&+\sum_{G\in\mathcal{G}}\sum_{\substack{F\in\mathcal{F}\\ F\subset G}}\sum_{\substack{H\in\mathcal{H}\\ H\subset F}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)\\
&+\sum_{F\in\mathcal{F}}\sum_{\substack{H\in\mathcal{H}\\ H\subset F}}\sum_{\substack{G\in\mathcal{G}\\ G\subset H}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)\\
&+\sum_{G\in\mathcal{G}}\sum_{\substack{H\in\mathcal{H}\\ H\subset G}}\sum_{\substack{F\in\mathcal{F}\\ F\subset H}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)\\
&+\sum_{H\in\mathcal{H}}\sum_{\substack{F\in\mathcal{F}\\ F\subset H}}\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)\\
&+\sum_{H\in\mathcal{H}}\sum_{\substack{G\in\mathcal{G}\\ G\subset H}}\sum_{\substack{F\in\mathcal{F}\\ F\subset G}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}(\langle f\rangle_Q^{\sigma_1})^\gamma(\langle g\rangle_Q^{\sigma_2})^\gamma\langle h\rangle_Q^w\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)\\
&:=I+I'+II+II'+III+III'.
\end{aligned}
$$

First we estimate I. We have

$$
\begin{aligned}
I&\lesssim\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^\gamma\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}(\langle g\rangle_G^{\sigma_2})^\gamma\sum_{\substack{H\in\mathcal{H}\\ H\subset G}}\langle h\rangle_H^w\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q)\\
&\le\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^\gamma\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}(\langle g\rangle_G^{\sigma_2})^\gamma\int\Big(\sum_{\substack{H\in\mathcal{H}\\ H\subset G}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma\mathbf{1}_Q\Big)\Big(\sup_{\substack{H'\in\mathcal{H}\\ \pi_{\mathcal{G}}(H')=G}}\langle h\rangle_{H'}^w\mathbf{1}_{H'}\Big)\mathrm{d}w\\
&\le\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^\gamma\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}(\langle g\rangle_G^{\sigma_2})^\gamma\Big\|\sum_{\substack{H\in\mathcal{H}\\ H\subset G}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma\mathbf{1}_Q\Big\|_{L^q(w)}\|\sup_{\substack{H'\in\mathcal{H}\\ \pi(H')=(F,G)}}\langle h\rangle_{H'}^w\mathbf{1}_{H'}\|_{L^{q'}(w)}\\
&\le2^{\frac{\gamma a}{p}}\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^\gamma\sum_{\substack{G\in\mathcal{G}\\ G\subset F}}(\langle g\rangle_G^{\sigma_2})^\gamma\Big(\sum_{\substack{H\in\mathcal{H}\\ \pi(H)=(F,G)}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\sigma_1(Q)\Big)^{\frac{\gamma}{p_1}}\\
&\times\Big(\sum_{\substack{H\in\mathcal{H}\\ \pi(H)=(F,G)}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\sigma_2(Q)\Big)^{\frac{\gamma}{p_2}}\Big(\sum_{\substack{H\in\mathcal{H}\\ \pi(H)=(F,G)}}(\langle h\rangle_H^w)^{q'}w(H)\Big)^{\frac{1}{q'}}.
\end{aligned}
$$

Since $\frac{\gamma}{p_1}+\frac{\gamma}{p_2}+\frac{1}{q'}=1$, by using Hölder's inequality twice we have

$$
\begin{aligned}
I &\lesssim 2^{\frac{\gamma a}{p}}\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle_F^{\sigma_1})^{p_1}\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}\sum_{\substack{H\in\mathcal{H}\\ \pi(H)=(F,G)}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\sigma_1(Q)\Big)^{\frac{\gamma}{p_1}}\\
&\times\Big(\sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}(\langle g\rangle_G^{\sigma_2})^{p_2}[\sigma_2]_{A_\infty}\sigma_2(G)\Big)^{\frac{\gamma}{p_2}}\Big(\sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}\sum_{\substack{H\in\mathcal{H}\\ \pi(H)=(F,G)}}(\langle h\rangle_H^{w})^{q'}w(H)\Big)^{\frac{1}{q'}}\\
&\overset{(2.2)}{\lesssim} 2^{\frac{\gamma a}{p}}[\sigma_1]_{A_\infty}^{\frac{\gamma}{p_1}}[\sigma_2]_{A_\infty}^{\frac{\gamma}{p_2}}\|f\|_{L^{p_1}(\sigma_1)}^{\gamma}\|g\|_{L^{p_2}(\sigma_2)}^{\gamma}\|h\|_{L^{q'}(w)}.
\end{aligned}
$$

It is obvious that I' can be estimated similarly. Next we estimate II. We have

$$
\begin{aligned}
&\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}w(Q)\\
&=\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}\langle w\rangle_Q|Q|\\
&=\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}(\langle w\rangle_Q)^{\frac{\gamma p_1'}{p}}(\langle\sigma_1\rangle_Q)^{\frac{p}{p_1'}\cdot\frac{\gamma p_1'}{p}}(\langle\sigma_2\rangle_Q)^{\frac{p}{p_2'}\cdot\frac{\gamma p_1'}{p}}\langle\sigma_2\rangle_Q^{\gamma-\frac{\gamma p_1'}{p_2'}}\langle w\rangle_Q^{1-\frac{\gamma p_1'}{p}}|Q|\\
&\lesssim 2^{\frac{\gamma p_1' a}{p}}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\langle\sigma_2\rangle_Q^{\gamma-\gamma\frac{p_1'}{p_2'}}\langle w\rangle_Q^{1-\frac{\gamma p_1'}{p}}|Q|
\end{aligned}
$$

Since $p_1=\max\{p_1,p_2,q'\}$ and $p>\gamma$, it is easy to check that

$$0\le\gamma-\frac{\gamma p_1'}{p_2'}<1,\ 0\le 1-\frac{\gamma p_1'}{p}<1,$$

and

$$\frac{1}{r}:=\gamma-\frac{\gamma p_1'}{p_2'}+1-\frac{\gamma p_1'}{p}<1.$$

Set

$$\frac{1}{s}:=\gamma-\frac{\gamma p_1'}{p_2'}+\frac{1-\frac{1}{r}}{2}.$$

Then

$$\frac{1}{s'}=1-\frac{\gamma p_1'}{p}+\frac{1-\frac{1}{r}}{2},$$

and therefore,

$$\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}}\langle\sigma_1\rangle_Q^{\gamma}\langle\sigma_2\rangle_Q^{\gamma}w(Q)\lesssim 2^{\frac{\gamma p_1' a}{p}}\int_G M(\sigma_2\mathbf{1}_G)^{\gamma-\gamma\frac{p_1'}{p_2'}}M(w\mathbf{1}_G)^{1-\frac{\gamma p_1'}{p}}dx$$

$$\leq 2^{\frac{\gamma p_1' a}{p}} \Big(\int_G M(\sigma_2 \mathbf{1}_G)^{s(\gamma-\gamma\frac{p_1'}{p_2'})} dx \Big)^{\frac{1}{s}} \Big(\int_G M(w\mathbf{1}_G)^{s'(1-\frac{\gamma p_1'}{p})} \Big)^{\frac{1}{s'}}$$

Before we give further estimate, we introduce the *Kolmogorov's inequality* (see for example [22]): Let $0 < p < q < \infty$, then there exists a constant $C = C_{p,q}$ such that for any locally integrable function f,

$$\|f\|_{L^p(Q,\frac{dx}{|Q|})} \leq C\|f\|_{L^{q,\infty}(Q,\frac{dx}{|Q|})}.$$

With this inequality in hand, we have

$$\frac{1}{|G|}\int_G M(w\mathbf{1}_G)^{s'(1-\frac{\gamma p_1'}{p})} dx \leq \|M(w\mathbf{1}_G)\|_{L^{1,\infty}(G,\frac{dx}{|G|})}^{s'(1-\frac{\gamma p_1'}{p})} \leq \langle w\rangle_G^{s'(1-\frac{\gamma p_1'}{p})},$$

and

$$\Big(\frac{1}{|G|}\int_G M(\sigma_2\mathbf{1}_G)^{s(\gamma-\gamma\frac{p_1'}{p_2'})} dx\Big) \leq \|M(\sigma_2\mathbf{1}_G)\|_{L^{1,\infty}(G,\frac{dx}{|G|})}^{s(\gamma-\gamma\frac{p_1'}{p_2'})} \leq \langle \sigma_2\rangle_G^{s(\gamma-\gamma\frac{p_1'}{p_2'})}.$$

Thus we get

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}} \langle\sigma_1\rangle_Q^\gamma \langle\sigma_2\rangle_Q^\gamma w(Q) &\lesssim 2^{\frac{\gamma p_1' a}{p}} \langle\sigma_2\rangle_G^{\gamma-\gamma\frac{p_1'}{p_2'}} \langle w\rangle_G^{1-\frac{\gamma p_1'}{p}} |G| \\ &\lesssim 2^{\frac{\gamma a}{p}} w(G)^{1-\frac{\gamma}{p}} \sigma_1(G)^{\frac{\gamma}{p_1}} \sigma_2(G)^{\frac{\gamma}{p_2}}.\end{aligned}$$

It follows that

$$\begin{aligned} II &\lesssim \sum_{F\in\mathcal{F}} (\langle f\rangle_F^{\sigma_1})^\gamma \sum_{\substack{H\in\mathcal{H}\\ H\subset F}} \langle h\rangle_H^w \sum_{\substack{G\in\mathcal{G}\\ G\subset H}} (\langle g\rangle_G^{\sigma_2})^\gamma \sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}} \langle\sigma_1\rangle_Q^\gamma \langle\sigma_2\rangle_Q^\gamma w(Q) \\ &\lesssim \sum_{F\in\mathcal{F}} (\langle f\rangle_F^{\sigma_1})^\gamma \sum_{\substack{H\in\mathcal{H}\\ \pi_{\mathcal{F}}(H)=F}} \langle h\rangle_H^w \sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{H}}(G)=H}} (\langle g\rangle_G^{\sigma_2})^\gamma 2^{\frac{\gamma a}{p}} w(G)^{1-\frac{\gamma}{p}} \sigma_1(G)^{\frac{\gamma}{p_1}} \sigma_2(G)^{\frac{\gamma}{p_2}} \\ &\lesssim 2^{\frac{\gamma a}{p}} [w]_{A_\infty}^{1-\frac{\gamma}{p}} [\sigma_1]_{A_\infty}^{\frac{\gamma}{p_1}} \|f\|_{L^{p_1}(\sigma_1)}^\gamma \|g\|_{L^{p_2}(\sigma_2)}^\gamma \|h\|_{L^{q'}(w)}, \end{aligned}$$

where again, the Hölder's inequality and (2.2) are used in the last step.

Now we estimate II'. By similar arguments as that in the above, we have

$$\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}} \langle\sigma_1\rangle_Q^\gamma \langle\sigma_2\rangle_Q^\gamma w(Q) \lesssim 2^{\frac{\gamma a}{p}} w(F)^{1-\frac{\gamma}{p}} \sigma_1(F)^{\frac{\gamma}{p_1}} \sigma_2(F)^{\frac{\gamma}{p_2}}.$$

Then it follows that

$$\begin{aligned} II' &\lesssim \sum_{G\in\mathcal{G}} (\langle g\rangle_G^{\sigma_2})^\gamma \sum_{\substack{H\in\mathcal{H}\\ H\subset G}} \langle h\rangle_H^w \sum_{\substack{F\in\mathcal{F}\\ F\subset H}} (\langle f\rangle_F^{\sigma_1})^\gamma \sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}} \langle\sigma_1\rangle_Q^\gamma \langle\sigma_2\rangle_Q^\gamma w(Q) \\ &\leq 2^{\frac{\gamma a}{p}} [w]_{A_\infty}^{1-\frac{\gamma}{p}} [\sigma_2]_{A_\infty}^{\frac{\gamma}{p_2}} \|f\|_{L^{p_1}(\sigma_1)}^\gamma \|g\|_{L^{p_2}(\sigma_2)}^\gamma \|h\|_{L^{q'}(w)}. \end{aligned}$$

III and III' can also be estimated similarly.

2.3. **The case $p > \gamma$ with $p_2 = \max\{p_1, p_2, q'\}$.** By symmetry, this case can be estimated similarly as that in the previous subsection.

2.4. **The case $p > \gamma$ with $q' = \max\{p_1, p_2, q'\}$.** Again, we can decompose the summation to $I + I' + II + II' + III + III'$. The estimates of I and I' have no differences with the previous case. Now we consider II. We have

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}} \langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q) &= \sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}} \langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma\langle w\rangle_Q|Q|\\ &= 2^a \sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}} \langle\sigma_1\rangle_Q^{\gamma-\frac{p}{p_1'}}\langle\sigma_2\rangle_Q^{\gamma-\frac{p}{p_2'}}|Q|\end{aligned}$$

Since $p > \gamma$ and $q' \geq \max\{p_1, p_2\}$, we have

$$\gamma - \frac{p}{p_1'} \geq 0,\ \gamma - \frac{p}{p_2'} \geq 0,$$

and

$$\gamma - \frac{p}{p_1'} + \gamma - \frac{p}{p_2'} = 2\gamma + 1 - 2p < 1.$$

Then follow the same arguments as that in the above, we get

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{S}_a\\ \pi(Q)=(F,G,H)}} \langle\sigma_1\rangle_Q^\gamma\langle\sigma_2\rangle_Q^\gamma w(Q) &\lesssim 2^a\langle\sigma_1\rangle_G^{\gamma-\frac{p}{p_1'}}\langle\sigma_2\rangle_G^{\gamma-\frac{p}{p_2'}}|G|\\ &\lesssim 2^{\frac{\gamma a}{p}}\Big(\langle w\rangle_G\langle\sigma_1\rangle_G^{\frac{p}{p_1'}}\langle\sigma_2\rangle_G^{\frac{p}{p_2'}}\Big)^{1-\frac{\gamma}{p}}\langle\sigma_1\rangle_G^{\gamma-\frac{p}{p_1'}}\langle\sigma_2\rangle_G^{\gamma-\frac{p}{p_2'}}|G|\\ &= 2^{\frac{\gamma a}{p}} w(G)^{1-\frac{\gamma}{p}}\sigma_1(G)^{\frac{\gamma}{p_1}}\sigma_2(G)^{\frac{\gamma}{p_2}}.\end{aligned}$$

Then follow the same arguments as the previous subsection we can get the desired conclusion. The estimates of II', III and III' can also be estimated similarly.

3. Applications

Theorem 1.1 has some new applications. It is obvious that if an operator reduced to $T_{p_0,\gamma,\mathcal{S}}$ for some p_0 and γ, then it is enough to apply Theorem 1.1 for those particular p_0 and γ. Thus, to find out the A_p-A_∞ estimates for *Multilinear square functions* (which were introduced and investigated in [6, 28, 29]), considering Proposition 4.2. of [4], it is enough to apply Theorem 1.1 for $T_{1,2,\mathcal{S}}$.

To observe the other application, we first recall the class of multilinear integral operator which is bounded on certain products of Lebesgue spaces on $\mathbb{R}^n$ where associated kernel satisfies some mild regularity condition which is weaker than the usual Hölder continuity of those in the class of multilinear Calderón-Zygmund singular integral operators. This class of the operators motivated from the recent works [3, 10, 14, 22, 25, 26, 27] and weighted bounds for such operators studied in [2] very recently. The main example of such operators is *Multilinear Fourier multipliers*. Now, to deduce the A_p-A_∞ estimates for the operators of such class, it is enough to apply Theorem 1.1 for $T_{p_0,1,\mathcal{S}}$ applying the main theorems of [2]. It is worth-mentioning that the A_p-A_∞ estimates for linear Fourier multipliers was unknown as well as other noted multilinear operators.

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